$$ds^2 = dx^2 + dy^2 + dz^2$$

$\lambda \in [a, b]$

(*) $$L(C) = \int_C ds$$

$$\frac{dx}{d\lambda}, \quad \frac{dy}{d\lambda}, \quad \frac{dz}{d\lambda}$$

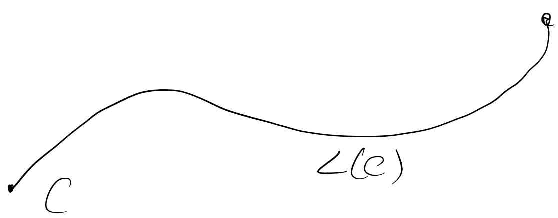
$$\left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2$$

(*) $$L(C) = \int_a^b d\lambda \, \frac{ds}{d\lambda} = \int_a^b d\lambda \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2}$$

$$ds^2 = g_{ij} \, dx^i dx^j$$

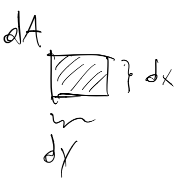
$$C : x^i(\lambda)$$

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$



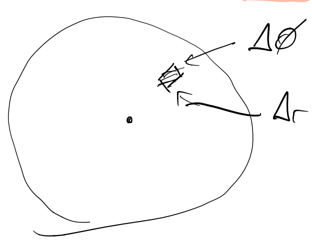
$$L(C) = \int_C ds = \int_a^b d\lambda \, \frac{ds}{d\lambda} = \int_a^b d\lambda \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

Areas / Volumes



$dA = dx \, dy$

$$ds^2 = \cancel{1} \, dr^2 + r^2 d\phi^2$$



$$dl_\phi = r \, d\phi$$

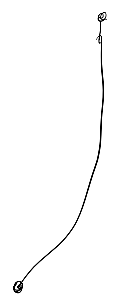
$$dl_r = dr$$

$$dA = dl_\phi \, dl_r = r\,dr\,d\phi$$

$g_{\mu\nu}$, $\quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

Flat space $\rightsquigarrow \; = -c^2 dt^2 + \underbrace{d\vec{x}^2}$

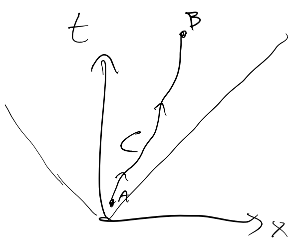
↳ proper lengths, areas, volumes, ...



$$\underline{-c^2 d\tau^2 = ds^2} = \underbrace{g_{\mu\nu}\, dx^\mu dx^\nu}_{<0}$$

$$\left(-(\dots)\, dt^2 \right)$$

static observer



$\underline{x^\mu(\lambda)}$

$$\Delta\tau = \int_C d\tau = \underbrace{\int_a^b d\lambda \sqrt{-\frac{1}{c^2}\, g_{\mu\nu}\, \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}$$

E.g.:

$g_{\mu\nu}$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\Longleftrightarrow \qquad g_{\mu\nu} \leftrightarrow \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$ds^2 = -dt^2 + x^2 dx^2 + xy\, \underline{dx\, dz} + dy^2$$

$$g_{\mu\nu} \leftrightarrow \begin{array}{c|c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \hline \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & x^2 & 0 & xy/2 \\ 0 & 0 & 1 & 0 \\ 0 & xy/2 & 0 & 0 \end{pmatrix} \end{array}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$g_{\mu\nu} \longleftrightarrow \begin{array}{c|cccc} & t & r & \theta & \phi \\ \hline t & -f(r) & 0 & 0 & 0 \\ r & 0 & g(r) & 0 & 0 \\ \theta & 0 & 0 & r^2 & 0 \\ \phi & 0 & 0 & 0 & r^2\sin^2\theta \end{array}$$